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In this paper, we suggest a modification of the Toeplitz approximation method for estimating frequencies of multiple sinusoids from covariance measurements. The method constructs a state-feedback matrix following a low-rank approximation of the Toeplitz covariance matrix via singular value decomposition. Ideally, the eigenvalues of this state-feedback matrix will be on the unit circle in the complex plane, and the angles that they make with the real axis will be equal to the unknown sinusoid frequencies. The modification proposed here exploits this prior knowledge of the modulus of the eigenvalues, and guarantees that even in the presence of noise, the eigenvalues of the estimated state-feedback matrix will lie on the unit circle.

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## AN IMPROVED TOEPLITZ APPROXIMATION METHOD<sup>1</sup>

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### ABSTRACT

In this paper, we suggest a modification of the Toeplitz approximation method for estimating frequencies of multiple sinusoids from covariance measurements. The method constructs a state-feedback matrix following a low-rank approximation of the Toeplitz covariance matrix via singular value decomposition. Ideally, the eigenvalues of this state-feedback matrix will be on the unit circle in the complex plane, and the angles that they make with the real axis will be equal to the unknown sinusoid frequencies. The modification proposed here exploits this prior knowledge of the modulus of the eigenvalues, and guarantees that even in the presence of noise, the eigenvalues of the estimated state-feedback matrix will lie on the unit circle.

### 1. INTRODUCTION

The problem of retrieving multiple sinusoids (with frequencies close to each other) from perturbed time-series or covariance information is of special interest in a vast range of signal-processing applications. Very often the covariance sequence may have to be estimated from time-series data, as in Doppler processing in radar. It is not uncommon, however, to encounter applications in which the (time-series) data are not measurable while the covariance information is directly available. Such situations arise in astronomical star bearing estimation, interference spectroscopy, and some sensor array applications.

In recent years, there has been a great deal of interest in model-based sinusoid retrieval. Models convert the non-linear problem of estimating the sinusoid frequencies into a simpler problem of estimating the parameters of a linear model [1]. The second step in all model-based methods is the extraction of the desired information (the frequencies) from the estimated model parameters [2]. Both steps are important for the overall success of a model-based method. Ill-conditioning at either step can adversely effect the overall performance of the method and should be avoided. The reliability of the first step depends on the estimation procedure, and that of the second step on the sensitivity of the desired information to the model parameters [3]. A popular model for the sinusoid retrieval problem is the linear prediction model first used by Prony in 1881.

$$y(t) = \sum_{k=1}^p a_k y(t-k)$$

whose parameters may be reliably estimated by the method of Tufts and Kumaresan [4]. The roots of the polynomial formed from these parameters are ideally expected to be on the unit circle in the complex plane, and the angles that they make with the real axis should equal the sinusoid frequencies.

### 2. STATE-SPACE REPRESENTATION

It turns out that the sinusoidal model is a very special case of the general linear rational model, and that just as there are alternate parameterizations of linear systems, there also are alternate parameter sets for the sinusoidal model as well. Just as there is a state-space representation for every realization of a linear, rational system, there is also a state-space representation for every realization of the sinusoidal model. The state-space representation of the special model for sinusoidal signals (frequencies:  $\omega_i$ ,  $i=1,2,\dots,n$ ) is:

$$\begin{aligned} x(k+1) &= Fx(k) \\ y(k) &= hx(k) \end{aligned}$$

where the order of the model  $p$  is twice the number of sinusoids, and the eigenvalues of  $F$  are of unit magnitude and equal  $e^{\pm j\omega_i}$ ,  $i=1,2,\dots,n$ . The sinusoidal signal  $y(t)$  is the model's zero-input response to some non-zero initial condition  $x(0)$ . In fact, we have

$$y(t) = hF^t x(0), \quad t \geq 0,$$

and the covariance  $r(m)$  of the sinusoidal signal satisfies

$$r(m) = hF^m P h^t \quad m \geq 0 \quad (1)$$

where  $P$  is the state-covariance matrix, and the superscript  $t$  denotes the Hermitian transpose.

The linear prediction model is a canonical realization of the above, with

$$x(t) = \begin{bmatrix} y(t-1) & y(t-2) & \dots & y(t-p) \end{bmatrix}^t.$$

$$F = \begin{bmatrix} a_1 & a_2 & \dots & a_p \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad h = \begin{bmatrix} a_1 & a_2 & \dots & a_p \end{bmatrix}.$$

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Another canonical realization which avoids the second step completely is

$$F = \text{diag}(e^{j\omega}), \quad h = (1, 1, \dots, 1).$$

However, the first step of estimating the model parameters becomes difficult for this realization. This non-uniqueness of the parameter triple  $(F, x(0), h)$  that characterizes the realization, allows one to choose a realization that makes both steps of the model-based method reliable. The state space parameters can be estimated from covariance data by using the factorizations derived below.

Using Eq. (1) for the covariance lags, and noting that the state covariance matrix  $P$  satisfies  $P = FPF^T$ , it can be shown that

$$r(-m) = hF^{-m}Ph^T. \quad (2)$$

Using Eqs. (1) and (2), the Toeplitz covariance matrix

$$R = \begin{bmatrix} r(0) & r(-1) & r(-2) & \dots & r(-n) \\ r(1) & r(0) & r(-1) & \dots & r(-n+1) \\ r(2) & r(1) & r(0) & \dots & r(-n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(n) & r(n-1) & r(n-2) & \dots & r(0) \end{bmatrix}$$

can be factorized as shown below.

$$R = \begin{bmatrix} h \\ hF \\ hF^2 \\ \vdots \\ hF^n \end{bmatrix} \begin{bmatrix} Ph^T & F^{-1}Ph^T & F^{-2}Ph^T & \dots & F^{-n}Ph^T \end{bmatrix}$$

$$R = \Theta \Gamma.$$

Both  $\Theta$  and  $\Gamma$  are full rank, and so the rank of  $R$  is equal to the model order  $p$ , which is twice the number of sinusoids [5]. Observe that the  $i^{\text{th}}$  row of  $\Theta$  is  $hF^{i-1}$ , and the  $i^{\text{th}}$  column of  $\Gamma$  is  $F^{-i+1}Ph^T$ , so that  $F$  may be obtained by solving the overdetermined system of equations

$$\Theta_1 F = \Theta_2 \quad (3)$$

where  $\Theta_1$  ( $\Theta_2$ ) is obtained from  $\Theta$  by deleting the last (first) row, or by solving

$$F^{-1}\Gamma_1 = \Gamma_2, \quad (4)$$

where  $\Gamma_1$  and  $\Gamma_2$  are defined in a manner similar to  $\Theta_1$  and  $\Theta_2$  respectively.

### 3. ORIGINAL TAM

The above discussion indicates how the sinusoid frequencies may be obtained from covariance data, when the information is exact. Let the singular value decomposition (SVD) of the Toeplitz matrix  $R$  (which is also the eigendecomposition when the covariances are exact) be  $R = U\Sigma V^T$ , where  $\Sigma$  contains only the non-zero singular values. Hence, the dimensions of diagonal matrix  $\Sigma$  will equal the rank of  $R$ , which is also the order of the model  $p$ . Different factorizations

$$\Theta = U\Sigma^{1/2}Q, \quad \Gamma = Q^{-1}\Sigma^{1/2}V^T$$

of the Toeplitz matrix are possible, and each will lead to a

different state-space realization of the order- $p$  sinusoidal model as

$$F = \Theta_1^L \Theta_2 = (\Gamma_2 \Gamma_1^R)^{-1}. \quad (5)$$

Here, the superscript  $L$  (similarly  $R$ ) denotes any left-inverse (or right-inverse). They exist because the matrices under consideration have full rank. The frequencies of the sinusoids may then be found as the angles of the eigenvalues of the  $F$  matrix.

Theoretically, the algorithm should work for any choice of  $Q$ . For instance, the choice  $Q = \Sigma^{1/2}V_1^T$ , where  $V_1$  denotes the matrix formed from the first  $p$  rows of  $V$ , will lead to the canonical realization with the linear prediction parameters. The so-called balanced choice  $Q = I$  is the preferred choice for reducing the sensitivity of the sinusoid frequencies to the model parameters.

It is well known in the numerical analysis literature that the eigenvalues of a normal matrix are least sensitive to perturbations in the matrix entries [6]. Recall that  $F$  is a normal matrix if it satisfies  $FF^T = F^T F$ . Since the state-feedback matrix of the sinusoidal model has eigenvalues of unit modulus, this amounts to requiring that the matrix be unitary. Note that for every realization of the sinusoidal model,  $F$  has eigenvalues on the unit circle, which is a necessary condition for it to be unitary. It is not a sufficient condition however, and not all realizations of the sinusoidal model have unitary state-feedback matrices; but it turns out that

**Theorem 1:** The  $F$  matrix obtained from any symmetric factorization ( $\Theta = \Gamma^T$ ) of the square covariance matrix  $R$  is unitary.

**Proof:** Let  $\Theta = \Gamma^T$ . Then  $\Theta_1 = \Gamma_1^T$  and  $\Theta_2 = \Gamma_2^T$ . Combining this with Eq. (5), we have  $F = (F^{-1})^T$  which implies  $F$  is unitary.  $\square$

The Toeplitz approximation method (TAM) of [7, 5] exploits these facts to reliably estimate the sinusoid frequencies from inexact covariances. Inexactness can be caused in practice, by a number of factors, additive noise in the data, errors in estimating the covariances, finite precision errors, and others. At first, TAM performs an SVD of  $R$ , and retains the  $p$  principal components (i.e., the  $p$  largest singular values and the corresponding singular vectors). SVD is preferred to eigendecomposition because in the presence of perturbations,  $R$  may not be non-negative definite. Let the singular vectors and singular values after the low-rank approximation be  $U, \Sigma$ , and  $V$ . Next, TAM picks  $\Theta = U\Sigma^{1/2}$ , and looks for an approximate solution to Eq. (3). The approximation criterion used is least-squares, and the TAM estimate is

$$\hat{F}_{\text{TAM}} = \Theta_1^\dagger \Theta_2, \quad (6)$$

where the superscript  $\dagger$  denotes the pseudo-inverse. The sinusoid frequency estimates are then, the angles of the eigenvalues of  $\hat{F}_{\text{TAM}}$ .

Theorem 1 indicates that in the noiseless situation, TAM's choice of factor  $\Theta$  as  $U\Sigma^{1/2}$  will make the  $F$ -matrix unitary. In the low SNR case as well, if there is an exact solution to Eq. (3), (which is the case when no low-rank approximation is made,  $R$  is non-negative definite and exactly equal to  $U\Sigma V^T$ , for instance) then the solution will

be unitary. But whenever a principal-components approximation is made and there is no exact solution to Eq. (3), the least-squares solution  $\hat{F}_{TAM}$  will generically not be unitary.

#### 4. IMPROVED TAM

The modification proposed in this section is the inclusion of the unitary constraint on  $F$  in the least-squares solution of Eq. (3). Such a constrained least-squares problem also arises in motion estimation and a simple analytical solution was derived in [8]. We include the problem and the solution here, but refer the reader to [8] for the derivation of the algorithm.

**Problem:** Given two real-valued  $n$  by  $p$  matrices  $\Theta_1$  and  $\Theta_2$ ,  $n \geq p$ , find  $p$  by  $p$  matrix  $F$  that minimizes

$$\text{Trace} \left[ (\Theta_1 F - \Theta_2)^T (\Theta_1 F - \Theta_2) \right],$$

subject to the constraint that  $F$  be unitary.

**Solution:** Calculate the  $p$  by  $p$  matrix

$$H = \Theta_2^T \Theta_1.$$

Let the SVD of  $H$  be  $U_H \Sigma_H V_H^T$ , where  $\Sigma_H$  includes all  $p$  singular values, even if they are zero. Then the solution to the constrained least-squares problem is

$$\hat{F}_{\text{unitary}} = V_H U_H^T. \quad (7)$$

We propose the use of this solution in place of Eq. (6) to guarantee that the estimated  $F$ -matrix is unitary and consequently does have eigenvalues on the unit circle. To reduce the effects of noise on the estimate of  $F$ , TAM's choice of  $\Theta$  is also slightly modified as follows. The complete algorithm is

**Improved TAM algorithm:**

1. Compute the SVD of  $R$

$$R = \sum_{k=1}^r u_k \sigma_k v_k^T,$$

where the rank  $r$  is at least as large as  $p$ , and the singular values are numbered in order of decreasing magnitude. Then, denote

$$U = [u_1 \cdots u_p], \quad \Sigma = \text{diag}(\sigma_k - \sigma_{p+1}; k=1, \dots, p);$$

and choose

$$\Theta = U \Sigma^{1/2}$$

2. Let the SVD of  $H = \Theta_2^T \Theta_1$  be  $U_H \Sigma_H V_H^T$ , where  $\Sigma_H$  includes all  $p$  singular values. Then the new estimate of the state-feedback matrix is

$$\hat{F}_{\text{unitary}} = V_H U_H^T.$$

A heuristic reasoning for subtracting the  $(p+1)$ -th singular value in the construction of  $\Theta$  is the following. When there is additive white noise in the data and the covariances are exact, all singular values of  $R$  get translated up by the noise variance, but the singular vectors are not effected. So, if  $\Theta$  is constructed from the  $p$  principal components, then, Eq. (3) will have an exact solution. But if  $\sigma_{p+1}$  is not subtracted off, the solution will not be unitary.

#### 5. SIMULATIONS

Our simulations have shown noticeable improvement in TAM's performance when exact covariances are available; but negligible improvement when covariances have to be estimated from low SNR data.

**Example 1.** In the first example, it is assumed that exact covariances are available, corresponding to a single sinusoid in an additive AR(1) process.

$$r(m) = \cos(2\pi f m) + \rho(0.5)^{|m|}$$

The improved TAM and the original TAM were used on covariances  $r(0), \dots, r(12)$ ; corresponding to different values of the sinusoid frequency  $f$ . The Toeplitz matrix used was of size  $13 \times 13$ , and a rank-2 approximation was employed in each algorithm. The simulations were performed in double precision on a VAX 780 using FORTRAN 77. The IMSL routines LSVDF, and EIGRF were used in the programs.

Table I gives the results for SNR 10 dB ( $\rho = 0.1$ ). Our results indicate that TAM fails to detect the sinusoid at frequencies around  $10^{-3}$  (i.e., both eigenvalues were real), and the improved TAM is able to resolve the two complex exponentials at much smaller frequencies, but appears to make large errors in the frequency estimate.

Table II gives the results for low SNR for a fixed frequency  $f=0.2$ . Both methods appear to fail at the same SNR in this simulation.

**Example 2.** In this example, data corresponding to two equi-amplitude real sinusoids in additive white noise was used as the starting point for the estimation schemes.

$$y(n) = A \cos(2\pi f_1 n) + A \cos(2\pi f_2 n + \phi) + w(n),$$

where  $\phi$  is a random variable uniformly distributed over  $[-\pi, +\pi]$ , and  $w(n)$  is zero-mean Gaussian white noise, independent of  $\phi$ . Only 48 consecutive observations from a single sample sequence are given; and the covariance lags have to be estimated by temporal averaging over the single record. For the experiment, IMSL routines GGLUBS and GGNML were used to generate  $\phi$  and  $w(n)$  respectively. The original TAM as well as the improved TAM were both employed to estimate the sinusoid frequencies from unbiased covariance estimates, using rank-2 approximations. The experiment was repeated 50 times using independent realizations of  $\phi$  and  $w(n)$ , and the arithmetic mean and standard deviation was computed for the 50 estimates of each frequency. The simulations were performed in single precision on a 60-bit CYBER 175 using FORTRAN 77.

Table III gives the results for SNR 10 dB,  $f_1 = 0.125$ ,  $f_2 = 0.135$ , and matrix size 32 by 32. Table IV gives the results for SNR 3 dB,  $f_1 = 0.125$ ,  $f_2 = 0.145$ , and matrix size 24 by 24. The results seem to indicate that any improvement over the original TAM is marginal in this example. A possible explanation is that TAM has achieved (or is very close to) the Cramer-Rao lower bound and cannot be further improved. Another possibility is that the state-feedback matrix estimated by the the original TAM is already close to unitary, even before imposing the constraint. Both possibilities are the subject of current investigation.

True $f$	0.20000	0.02000	0.00200	0.00100	0.00050	0.00020
Original TAM	0.19979	0.02004	0.00184	0.00000	0.00000	0.00000
Improved TAM	0.19980	0.01988	0.00206	0.00158	0.00146	0.00143

Table I (SNR = 10 dB)

SNR	10 dB	0 dB	-2 dB	-4 dB	-5 dB
Original TAM estimate	0.19979	0.19767	0.19583	0.18968	0.16661
Improved TAM estimate	0.19980	0.19781	0.19607	0.19000	0.16362

Table II ( $f = 0.2$ )

		$\hat{f}_1$	$\hat{f}_2$
	Mean	0.12848	0.13761
Orig. TAM			
	Std. dev.	$3.94 \times 10^{-7}$	$5.35 \times 10^{-6}$
	Mean	0.12854	0.13743
Impr. TAM			
	Std. dev.	$3.83 \times 10^{-7}$	$5.21 \times 10^{-6}$

Table III ( $f_1=0.125, f_2=0.135$ )

		$\hat{f}_1$	$\hat{f}_2$
	Mean	0.12713	0.16271
Orig. TAM			
	Std. dev.	$4.34 \times 10^{-5}$	$7.42 \times 10^{-3}$
	Mean	0.12439	0.15275
Impr. TAM			
	Std. dev.	$3.64 \times 10^{-4}$	$3.71 \times 10^{-3}$

Table IV ( $f_1=0.125, f_2=0.145$ )

## 6. SUMMARY

In summary, this paper examined the imposition of a unitary constraint on the state-feedback matrix constructed by the Toeplitz approximation method for estimating the frequencies of multiple sinusoids. The constraint is a means of incorporating given knowledge about the sinusoidal model, namely that the eigenvalues of the state-feedback matrix are on the unit circle. This information can also be partially incorporated in linear-prediction methods for the same problem. It has been previously suggested that a symmetry constraint be imposed on the estimated linear prediction polynomial, because it is known that the polynomial must necessarily be symmetric to have roots on the unit circle. However, this symmetry condition is only necessary and not sufficient for ensuring that the roots have unit modulus. For instance, a symmetric polynomial may have roots at  $\alpha$  and  $\frac{1}{\alpha}$  instead of unit-modulus roots. In contrast, the constraint studied here is both necessary and sufficient as demonstrated by Theorem 1.

Preliminary numerical simulations seem to indicate that an improvement is possible, when covariances are directly measured. In every example, the modified TAM was at least as good as the original TAM, both in resolution and in estimation error. The modification proposed here may be particularly useful in applications where the covariances are obtained by (pseudo-) ensemble averaging instead of temporal averaging. Further experimentation and analysis is needed to determine the kinds of problems for which the extra computational expense pays off in marked improvement.

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